On topologizable and non-topologizable groups

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Abstract

A group G is called hereditarily non-topologizable if, for every $H \leq G$, no quotient of H admits a non-discrete Hausdorff topology. We construct first examples of infinite hereditarily non-topologizable groups. This allows us to prove that c-compactness does not imply compactness for topological groups. We also answer several other open questions about c-compact groups asked by Dikranjan and Uspenskij. On the other hand, we suggest a method of constructing topologizable groups based on generic properties in the space of marked k-generated groups. As an application, we show that there exist non-discrete quasi-cyclic groups of finite exponent; this answers a question of Morris and Obraztsov.

1 Introduction

Throughout this paper, we always assume topological groups and spaces to be Hausdorff. A well known Kuratowski–Mrowka theorem states that a topological space X is compact if and only if, for any topological space Y, the projection $\pi_Y : X \times Y \to Y$ is closed. Motivated by this theorem, Dikranjan and Uspenskij [6] call a topological group X *categorically compact* (or *c-compact* for brevity) if, for every topological group Y, the image of every closed subgroup of $X \times Y$ under the projection $\pi_Y : X \times Y \to Y$ is closed in Y.

Obviously every compact group is c-compact, while the converse was open until now even for discrete groups. More precisely, the following questions were asked in [6, Question 1.2 and Question 5.2] (see also [24, Problem 31 (i) and Question 34]).

Problem 1.1.

(a) *Is every c-compact group compact?*

(b) *Is every discrete c-compact group finite (finitely generated, of finite exponent, countable)?*

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These questions have received considerable attention in the recent years. A complete survey of recent results can be found in the book [12], which is devoted to these problems. Until now, only results in the affirmative direction were known. For example, the answer to (a) is known to be positive for solvable groups and connected locally compact groups [6]. Note also that every discrete c-compact group is necessarily a torsion group by [6, Theorem 5.3].

Our first goal is to show that the answer to all parts of Problem 1.1 is, in fact, negative. Our approach is based on a sufficient condition for c-compactness suggested in [6], which leads to the notion of a hereditarily non-topologizable group introduced by Lukács [13].

Recall that an abstract group is called *topologizable* if it admits a non-discrete Hausdorff group topology, and *non-topologizable* otherwise. In 1946, A. A. Markov [15] asked whether there exist non-topologizable infinite groups and the problem remained open until late 70's. In [25], Shelah constructed first (uncountable) examples using the Continuum Hypothesis. Later Hesse [10] showed that the use of the CH in Shelah's proof can be avoided. The affirmative answer to the Markov's question for countable groups was obtained by the second author in [20] (see also [18, Theorem 31.5]); the proof uses the group constructed by Adjan in [1] and is essentially elementary modulo the main theorem of [1]. Since then many other examples of non-topologizable groups have been found (see, for example, [11]).

A group G is called *hereditarily non-topologizable* if for every $H \leq G$ and every $N \lhd H$, the quotient group H/N is non-topologizable. It is easy to prove that every hereditarily non-topologizable group is c-compact with respect to the discrete topology (see [6, Corollary 5.4]); moreover, a countable group is hereditarily non-topologizable if and only if it is c-compact with respect to the discrete topology [6, Theorem 5.5].

Using techniques developed in [18] we prove the following result (see Theorem 2.5), which completely solves Problem 1.1.

Theorem 1.2. *There exist hereditarily non-topologizable (and hence c-compact with respect to the discrete topology) groups G, H, I, and J such that:*

(a) *G is infinite, finitely generated, and of bounded exponent;*

(b) *H is finitely generated and of unbounded exponent;*

(c) *I is countable, but not finitely generated;*

(d) *J is uncountable.*

On the other hand, it is worth noting that neither of the groups constructed in [11, 20, 25] is c-compact (see Remark 2.6).

The finitely generated groups G and H from Theorem 1.2 are the so-called Tarski Monsters, i.e., infinite simple groups with all proper subgroups finite cyclic. First examples of such groups were constructed by the second author in [21]. Clearly every non-topologizable Tarski Monster is hereditarily non-topologizable. This raises the natural question of whether a Tarski Monster can be topologized. The standard way of defining a non-discrete topology using chains of subgroups obviously fails for Tarski Monsters. Moreover, most groups which are known to be topologizable, such as infinite residually finite groups, infinite locally finite groups [2], or

groups containing infinite normal solvable subgroups [10], are located on the opposite side of the group-theoretic universe.

The first (and the only known) examples of topologizable Tarski Monsters were constructed by Morris and Obraztsov in [16] using methods from [18]. An essential feature of the Morris–Obraztsov construction is that their groups have unbounded exponent and for finite exponent their method of defining a non-discrete topology seems to fail. This motivated the following.

Question 1.3. [16, Question 3] *Does there exist a topologizable quasi-finite group of finite exponent?*

Recall that a group is *quasi-finite* if all its proper subgroups are finite and is of *finite exponent* n if $g^n = 1$ for some positive integer n and every $g \in G$. In this paper we answer the Morris–Obraztsov's question affirmatively.

Theorem 1.4. *For every sufficiently large odd $n \in \mathbb{N}$ there exists a topologizable Tarski Monster of exponent n.*

Our proof of Theorem 1.4 utilizes the notion of a generic property in a topological space. Recall that a subset S of a topological space X is called a *G_δ set* if S is an intersection of a countable collection of open sets. Further one says that a *generic element of X has a certain property P* (or P is *generic* in X) if P holds for every x from some dense G_δ subset of X. The Baire Category Theorem implies that in a complete metric space the intersection of any countable collection of dense G_δ sets is again dense G_δ. Thus we can combine generic properties: if every property from a countable collection $\{P_1, P_2, \ldots\}$ is generic, then so is the whole collection (i.e., the conjunction of $P_1, P_2, \ldots$). In many situations this approach is useful for proving the *existence* of elements of X simultaneously satisfying $P_1, P_2, \ldots$.

To implement this idea we consider the *space of marked k-generated groups*, $\mathcal{G}_k$, which is a compact totally disconnected metric space consisting of all k-generated groups with fixed generating sets. For the precise definition we refer to Section 3. The following observation is crucial for our proof of Theorem 1.4.

Proposition 1.5. *For every $k \in \mathbb{N}$, the following subsets of $\mathcal{G}_k$ are G_δ:*

(a) *the set of all topologizable groups;*

(b) *the set of all Tarski Monsters of any fixed finite exponent.*

Using methods from the book [18], for every sufficiently large odd $n \in \mathbb{N}$ we construct a compact nonempty subset $\mathcal{T} \subseteq \mathcal{G}_2$ consisting of groups of exponent n such that $\mathcal{T}$ contains a dense subset of topologizable groups and a dense subset of Tarski Monsters. Then by Proposition 1.5 the properties of being topologizable and being a Tarski Monster are generic in $\mathcal{T}$. Hence topologizable Tarski Monsters (of exponent n) are generic in $\mathcal{T}$. In particular, they exist.

All Tarski Monsters discussed above, as well as many other groups with "exotic" properties, are limits of hyperbolic groups. It is not difficult to see that every infinite hyperbolic group is topologizable, but a much weaker condition also make a group topologizable; namely, in the last

section we observe that being topologizable is a generic property among limits of "hyperbolic-like" groups. More precisely, we consider the class of groups with non-degenerate hyperbolically embedded subgroups, which was introduced in [5]; here *non-degenerate* simply means infinite and proper. This class contains all non-elementary hyperbolic groups, non-elementary relatively hyperbolic groups with proper peripheral subgroups (e.g., all non-trivial free products other than $\mathbb{Z}_2 * \mathbb{Z}_2$), mapping class groups of surfaces of genus > 1, $Out(F_n)$ for $n \geq 2$, and many other interesting examples. For the definition and more details we refer to [5].

Given a subset $S \subseteq \mathcal{G}_k$, we denote by $\overline{S}$ its closure in $\mathcal{G}_k$.

Theorem 1.6. *Let $\mathcal{S} \subseteq \mathcal{G}_k$ be a subset consisting of (marked k-generated) groups with non-degenerate hyperbolically embedded subgroups. Then being topologizable is a generic property in the set $\overline{\mathcal{S}}$.*

The proof of Theorem 1.6 is accomplished by proving that every group with non-degenerate hyperbolically embedded subgroups is topologizable (see Lemma 5.1). Then Proposition 1.5 yields the claim. Finally we sketch a possible application of Theorem 1.6 to constructing non-discrete groups with all nontrivial elements conjugate (for details and motivation see Section 5).

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2 Hereditarily non-topologizable groups

Recall that a subset V of a group G is called *elementary algebraic* if there exist $a_1, \ldots, a_k \in G$ and $\varepsilon_1, \ldots, \varepsilon_k \in \mathbb{Z}$ such that V is the set of all solutions of the equation $a_0 x^{\varepsilon_1} a_1 x^{\varepsilon_2} \cdots a_k x^{\varepsilon_k} = 1$ in G.

Lemma 2.1 (A.A. Markov [15])**.** *A countable group G is non-topologizable if and only if $G \setminus \{1\}$ is a finite union of elementary algebraic sets.*

We say that a group G is given by a *presentation over a free product* $G_0 * G_1 * \ldots$ if G is presented in the form

$$G = (G_0 * G_1 * \ldots) / \langle\langle R_1, R_2, \ldots \rangle\rangle .$$

We are interested in presentations satisfying *condition R* introduced in [18]. For the purpose of this paper it suffices to know that condition R is a rather technical requirement on the additional relators $R_1, R_2, \ldots$ making it possible to construct groups with various unusual properties. The book [18] contains many such examples, one of which is the following theorem.

Obraztsov's Theorem. [17] (see also [18, Theorem 35.1])*. For any sufficiently large odd number n_0 and any countable (finite or infinite) family of nontrivial countable groups $G_0, G_1, \ldots$ without elements of order 2, there exists an infinite simple group $O(G_0, G_1, \ldots)$ containing all G_i as distinct maximal subgroups such that any proper subgroup of $O(G_0, G_1, \ldots)$ is either cyclic of order dividing n_0 or conjugate to a subgroup of some G_i. Any two maximal subgroups of $O(G_0, G_1, \ldots)$ intersect trivially, and this group has a presentation with condition* R *over the free product $G_0 * G_1 * \ldots$ if there are at least two groups G_i.*

Lemma 2.2. *If a group U has a presentation with condition* R *over a free product $G_0 * G_1 * \dots$, where G_i are countable nontrivial groups without elements of order 2, then for any elements $g_0 \in G_0 \setminus \{1\}$ and $u \in U \setminus G_0$, the element $g_0 g_0^u$ is not conjugate to any element of any group G_i.*

Henceforth, g^h means $h^{-1}gh$ if g and h are elements of a group.

Proof. Suppose that this element is conjugate to an element $h \in G_i$. Let $X \in G_0 * G_1 * \dots$ be a word (in the alphabet $G_0 \cup G_1 \cup \dots$) representing u.

It follows that there is a diagram of conjugacy of the word $g_0 X^{-1} g_0 X$ and the letter h. One can identify the subpaths of the boundary labeled by X and X^{-1} and obtain a diagram Δ_0 on a sphere with 3 holes. Its boundary components p_1, p_2 and q are labeled (e.g., in clockwise manner) by the letters g_0, g_0, and h. A simple path x labeled by X connects the origins of the paths p_1 and p_2.

If Δ_0 is not a reduced diagram, then one can make reductions described in Section 13 of [18] and obtain a reduced diagram Δ with the same boundary labels. Moreover, there is a simple path in Δ connecting the origins of p_1 and p_2, whose label is equal to X in U.

We obtain a reduced diagram Δ on a sphere with 3 holes, but the length of every its boundary component is equal to 1 (in the metric of diagrams over free products). It follows that the rank of Δ is 0. Indeed, Lemmas 33.3 – 34.2 [18] say that one can extend the theory of presentations with the condition R to the presentations over free products; and so if $r(\Delta) > 0$, then the length of one of the boundary components of Δ must be $> \varepsilon n > 1$ by Theorem 22.2. Here ε and n are some parameters from [18] satisfying $n \gg \frac{1}{\varepsilon} \gg 1$; their exact values are not essential for us. (For more details about parameters see Section 4.)

Thus the word X is equal in U to a word X' such that the element $g_0 g_0^{X'}$ is conjugate to an element of G_i in the free product $G_0 * G_1 \dots$. But this can happen only if X' represents an element of G_0. Hence $u \in G_0$. This contradiction completes the proof. $\square$

Theorem 2.3. *For any sufficiently large integer n_0 and any countable nonempty family of nontrivial countable groups $G_1, G_2, \dots$ without elements of order 2, there exists an infinite simple group $O'(G_1, G_2, \dots)$ containing all G_i as maximal and distinct subgroups such that any proper subgroup of $O'(G_1, G_2, \dots)$ is either cyclic of order at most n_0 or conjugate to a subgroup of some G_i. Any two distinct maximal subgroups of $O'(G_1, G_2, \dots)$ intersect trivially. The group $O'(G_1, G_2, \dots)$ is non-topologizable and, moreover, in this group, there exists an equation with one unknown having precisely one non-solution.*

Proof. Let us take a sufficiently large odd number n_0 as in Obraztsov's Theorem and a finite cyclic group $G_0 = \langle g \rangle$ of odd order coprime to n_0 (of order three, for instance). Now, we put

$$O'(G_1, G_2, \dots) = O(G_0, G_1, G_2, \dots).$$

This group is non-topologizable (by Lemma 2.1), because the set of non-solutions of the equation $(gg^x)^{n_0} = 1$ in this group is finite and nonempty. Indeed, this set contains the identity element and is contained in the finite group G_0, because if an element u does not lie in G_0,

then by Lemma 2.2 the element gg^u is conjugate to no elements of groups G_i and, therefore, generates a cyclic subgroup of order dividing n_0 by Obraztsov's theorem.

It is easy to construct an equation with precisely one non-solution in $O'(G_1, G_2, \dots)$. Indeed, take any element $v \in O'(G_1, G_2, \dots) \setminus G_0$ and consider the equation

$$[(gg^x)^{n_0}, (g^v g^{vx})^{n_0}] = 1.$$

The identity element is not a solution to this equation, because otherwise we would find a metabelian subgroup (either $\langle g, g^v\rangle$ or $\langle g, v\rangle$) containing $G_0 = \langle g\rangle$ as a proper subgroup (that contradicts the maximality of G_0 and the simplicity of the whole group). Indeed, the equality $[g^{2n_0}, g^{2n_0 v}] = 1$ implies $[g, g^v] = 1$, because the order of g is coprime to $2n_0$. Thus, the group $\langle g, g^v\rangle$ is abelian; if $\langle g, g^v\rangle \neq \langle g\rangle$ then we have found a desired subgroup, else the group $\langle g, v\rangle$ is metabelian and contains $\langle g\rangle = G_0$ as a proper subgroup.

All nonidentity elements of $O'(G_1, G_2, \dots)$ are solutions of the equation, because the first argument of the commutator is 1 for all x not lying in G_0, as explained above; the second argument of the commutator is 1 for all x not lying in $(G_0)^v$ (by the same reasons); and $G_0 \cap (G_0)^v = \{1\}$, since both subgroups are maximal and distinct (see Obraztsov's theorem).

The remaining properties of $O'(G_1, G_2, \dots)$ follow directly from Obraztsov's theorem. $\square$

Remark 2.4. It is known that

- any group embeds into a non-topologizable group [26];
- there exists a (non-topologizable) torsion-free group of any infinite cardinality such that some equation has exactly one non-solution in this group [11];
- there exists an infinite (non-topologizable) group naturally isomorphic to its automorphism group such that some equation has exactly one non-solution in this group [27].

Theorem 2.5. *There exist infinite hereditarily non-topologizable simple torsion groups G, H, I, and J such that:*

(a) *G is 2-generated, quasi-cyclic, and of bounded exponent;*

(b) *H is 2-generated, quasi-cyclic, and of unbounded exponent;*

(c) *I is not finitely generated, countable, and of bounded exponent;*

(d) *J is uncountable and of bounded exponent.*

Proof. Let n_0 be a sufficiently large odd integer as in Theorem 2.3. Let

$$G = O'(\mathbb{Z}_{n_0}), \qquad H = O'(\mathbb{Z}_{n_0}, \mathbb{Z}_{n_0+2}, \mathbb{Z}_{n_0+4}, \dots), \qquad I = \bigcup_{i=0}^{\infty} G^i,$$

where $G^0 = G_0 = \langle g\rangle$ is a finite cyclic group of odd order coprime to n_0, and $G^{i+1} = O(G^i, \mathbb{Z}_{n_0})$ for $i > 0$.

The uncountable group J (of the first uncountable cardinality) is constructed similarly to I but using the transfinite induction (up to the first uncountable ordinal).

The groups G and H are quasi-cyclic and simple by Theorem 2.3. The groups I and J are simple, because they are unions of increasing chains of simple subgroups.

All four groups are non-topologizable. For G and H, this follows directly from Theorem 2.3; I and J are non-topologizable, since the set of non-solutions of the equation $(gg^x)^{n_0} = 1$ is nonempty and finite (it is contained in G_0) by Lemma 2.2.

The groups I and J contain no proper infinite subgroups except subgroups conjugate to simple non-topologizable groups G^i that implies hereditary non-topologizability. For both I and J, this is shown in the proof of Theorem 35.2 from [18]. □

Remark 2.6. Note that neither of the groups constructed in [11, 20, 25] is c-compact. Indeed it is immediate from the definition that c-compactness is preserved by taking closed subgroups (i.e., any subgroups in the discrete case). Recall that a discrete countable group is c-compact if and only if it is hereditarily non-topologizable. Since (discrete) groups from [25] and [11] contain infinite cyclic subgroups, they are not c-compact.

The countable non-topologizable group constructed in [20] is also not hereditarily non-topologizable, since it has the free Burnside group $B(m,n)$ with $m \geq 2$ generators and of large odd exponent n as a quotient. The latter group admits a non-discrete topology defined by a nested chain of normal subgroup. This can be extracted from [18, Theorem 39.3]; for $m = 2$ this also follows from Corollary 4.8 applied to $J = \varnothing$. Alternatively one can argue as follows. By [18, Theorem 39.1] the group $B(m,n)$ contains $B(\infty,n)$. Passing to the abelianization we obtain a countably infinite sum of copies of $\mathbb{Z}/n\mathbb{Z}$, which is obviously topologizable.

3 The space of marked groups and G_δ sets

Let F_k be the free group of rank k with basis $X = \{x_1, \ldots, x_k\}$ and let $\mathcal{G}_k$ denote the set of all normal subgroups of F_k. Given $M, N \lhd F_k$, let

$$\mathrm{d}(M,N) = \begin{cases} \min\left\{ \frac{1}{|w|} \;\middle|\; w \in N \bigtriangleup M \right\}, & \text{if } M \neq N \\ 0, & \text{if } M = N, \end{cases}$$

where $|\cdot|$ denotes the word length with respect to the generating set X. It is easy to see that $(\mathcal{G}_k, \mathrm{d})$ is a compact Hausdorff totally disconnected (ultra)metric space [9].

Note that one can naturally identify $\mathcal{G}_k$ with the set of all *marked k-generated groups*, i.e., pairs $(G, (x_1, \ldots, x_k))$, where G is a group and $(x_1, \ldots, x_k)$ is a generating k-tuple of G. (By abuse of notation, we keep the same notation for the generators $x_1, \ldots, x_k$ of F_k and their images in G.) For this reason the space $\mathcal{G}_k$ with the metric defined above is called the *space of marked groups with k generators*. For brevity, we simply call elements of $\mathcal{G}_k$ groups instead of marked k-generated groups.

Let $\mathcal{L}_k$ be the first order language that contains the standard group operations $\cdot$, $^{-1}$, the constant symbol 1, and constant symbols $x_1, \ldots, x_k$. Every element $(G, (x_1, \ldots, x_k)) \in \mathcal{G}_k$ can be naturally thought of as an $\mathcal{L}_k$-structure.

The following lemma is obvious.

Lemma 3.1. *Let w be a word in the alphabet $X \cup X^{-1}$. Then for every $k \in \mathbb{N}$, the set of groups in $\mathcal{G}_k$ satisfying $w = 1$ (or $w \neq 1$) is clopen.*

Proof. If $w = 1$ (or $w \neq 1$) in a group $(G, (x_1, \ldots, x_k)) \in \mathcal{G}_k$ and w has length r, then $w = 1$ (respectively, $w \neq 1$) in every other group $(H, (x_1, \ldots, x_k)) \in \mathcal{G}_k$ such that $\mathrm{d}(G, H) < 1/r$. Thus the set of groups satisfying $w = 1$ (respectively, $w \neq 1$) is open and the claim of the lemma follows. $\square$

Recall that a sentence in a first order language is called an $\forall\exists$-*sentence* if it has the form

$$\forall a_1 \ldots \forall a_m \exists b_1 \ldots \exists b_n \, \Phi(a_1, \ldots, a_m, b_1, \ldots, b_n), \tag{1}$$

where $\Phi(a_1, \ldots, a_m, b_1, \ldots, b_n)$ is a quantifier-free formula. If such a sentence only contains existential (respectively, universal) quantifiers, it is called *existential* (respectively, *universal*). We say that a subset $\mathcal{S} \subseteq \mathcal{G}_k$ is $\forall\exists$-*definable* if there exists an $\forall\exists$-sentence Σ in $\mathcal{L}_k$ such that

$$\mathcal{S} = \{P \in \mathcal{G}_k \mid P \models \Sigma\},$$

i.e., $\mathcal{S}$ is exactly the set of all elements of $\mathcal{G}_k$ satisfying Σ. Similarly we define *existentially definable* and *universally definable* subsets.

Observe that if $(G, (x_1, \ldots, x_k)) \in \mathcal{G}_k$, then we know that $x_1, \ldots, x_k$ generate G. This allows us to use the following quantifier elimination procedure. Let $R(u)$ be a (not necessarily first order) property of marked k-generated groups which depends on some parameter u interpreted as a group element. Enumerate all words $\{w_1, w_2, \ldots\}$ in the alphabet $X \cup X^{-1}$. Then we obviously have

$$\{P \in \mathcal{G}_k \mid P \models \forall u \, R(u)\} = \bigcap_{i=1}^{\infty} \{P \in \mathcal{G}_k \mid P \models R(w_i)\} \tag{2}$$

and

$$\{P \in \mathcal{G}_k \mid P \models \exists u \, R(u)\} = \bigcup_{i=1}^{\infty} \{P \in \mathcal{G}_k \mid P \models R(w_i)\}. \tag{3}$$

The first part of the following lemma is well-known although we were unable to find an exact reference.

Proposition 3.2. (a) [Folklore] *Every existentially defined subset of $\mathcal{G}_k$ is open.*

(b) *Every $\forall\exists$-definable subset of $\mathcal{G}_k$ is a G_δ set.*

Proof. Every existential sentence is equivalent to a sentence

$$\exists b_1 \cdots \exists b_n \, \Phi_1(b_1, \ldots, b_n) \vee \ldots \vee \Phi_q(b_1, \ldots, b_n), \tag{4}$$

such that each Φ_i is a system of equations and inequations of the form $w = 1$ (respectively, $w \neq 1$), where w is a word in the alphabet $\{x_1^{\pm 1}, \ldots, x_k^{\pm 1}\} \cup \{b_1^{\pm 1}, \ldots, b_n^{\pm 1}\}$. Thus the first claim follows from Lemma 3.1 and the quantifier elimination (3) applied to all quantifiers in (4). To prove (b) we have to eliminate all universal quantifiers in (1) according to (2) and apply (a). $\square$

It would be interesting to find other sufficient conditions in the spirit of [14] and [4] for a (not necessarily first order) sentence to define a G_δ subset of $\mathcal{G}_k$. In particular, we ask the following.

Question 3.3. *Which second order sentences define G_δ subsets of $\mathcal{G}_k$?*

The next proposition provides some particular non-trivial examples of G_δ subsets of $\mathcal{G}_k$, which are relevant to our paper. Recall that by a *Tarski Monster* we mean a finitely generated infinite simple group with all proper subgroups finite cyclic.

Proposition 3.4. *For every $k \in \mathbb{N}$, the following subsets of $\mathcal{G}_k$ are G_δ:*

(a) *The set of all topologizable groups.*

(b) *The set of all infinite groups.*

(c) *The set of all groups satisfying a given identity.*

(d) *The set of all simple groups.*

(e) *The set of Tarski Monsters of any fixed finite exponent.*

(f) *The set of groups with all non-trivial elements conjugate.*

Proof. Let $E = \{E_1, E_2, \ldots\}$ denote the set of all finite collections of equations over the free group F_k with one unknown. Given an element $(G, (x_1, \ldots, x_k)) \in \mathcal{G}_k$ (i.e., an epimorphism $\varepsilon \colon F_k \to G$), we can think of each E_n as a collection of equation over G by projecting all coefficients to G via ε. Consider the following condition:

$\mathbf{C}_n$: *Some nontrivial element of G satisfies neither of the equations from E_n or* 1 *satisfies at least one of the equations from E_n.*

Clearly every $\mathbf{C}_n$ can be expressed by an existential formula in $\mathcal{L}_k$. Hence the set $\mathcal{C}_n$ of elements of $\mathcal{G}_k$ satisfying $\mathbf{C}_n$ is open for every n by Proposition 3.2. By Lemma 2.1, the set of all topologizable groups in $\mathcal{G}_k$ coincides with $\bigcap_{n\in\mathbb{N}} \mathcal{C}_n$ and hence it is a G_δ set by definition.

To prove (b) we first observe that the set of all groups of order $\leq m$ in $\mathcal{G}_k$ is finite, hence the set $\mathcal{I}_m$ of groups having more than m elements is open. Consequently, the set of all infinite groups is a G_δ set being the intersection of all $\mathcal{I}_m$.

Part (c) follows from part (b) of Proposition 3.2 and the obvious fact that the subset of $\mathcal{G}_k$ consisting of groups satisfying a given identity can be defined by a universal sentence.

Let us prove (d). Fix some word w in $X \cup X^{-1}$ and enumerate all words $\{u_1, u_2, \ldots\}$ in the normal closure of w in F_k. Observe that the property

$\mathbf{D}_w$: *The normal subgroup of G generated by w is trivial or coincides with G*

can be expressed by the (infinite) disjunction of formulas

$$x_1 = u_{i_1} \,\&\, \ldots \,\&\, x_k = u_{i_k} \tag{5}$$

for $\{i_1, \ldots, i_k\} \in \mathbb{N}^k$ and $w = 1$. Hence the set $\mathcal{D}_w$ of elements of $\mathcal{G}_k$ satisfying $\mathbf{D}_w$ is the union of open subsets of $\mathcal{G}_k$ by Lemma 3.1. Consequently $\mathcal{D}_w$ is open. It is easy to check that a group $(G, (x_1, \ldots, x_k)) \in \mathcal{G}_k$ is simple if and only if it belongs to $\bigcap_{w \in F_k} \mathcal{D}_w$. Thus we obtain (d).

The proof of (e) is similar. Fix some $n \in \mathbb{N}$. For two words u, v in $X \cup X^{-1}$, consider the set $\mathcal{S}_{u,v}$ of all $(G, (x_1, \ldots, x_k)) \in \mathcal{G}_k$ such that the subgroup of G generated by $\{u, v\}$ is contained in a cyclic subgroup of order dividing n. It is easy to see that $\mathcal{S}_{u,v}$ can be defined by an existential formula in $\mathcal{L}_k$. E.g., for $n = 2$ the following formula works:

$$\exists z \; (z^2 = 1 \,\&\, ((u = 1 \,\&\, v = 1) \vee (u = 1 \,\&\, v = z) \vee (u = z \,\&\, v = 1) \vee (u = z \,\&\, v = z))).$$

Thus $\mathcal{S}_{u,v}$ is open in $\mathcal{G}_k$.

Further let $\{w_1, w_2, \ldots\}$ be the set of all elements of the subgroup of F_k generated by u and v. For any $(i_1, \ldots, i_k) \in \mathbb{N}^k$, denote by $\mathcal{R}_{u,v,i_1,\ldots,i_k}$ the set of all elements of $\mathcal{G}_k$ satisfying

$$x_1 = w_{i_1} \,\&\, \ldots \,\&\, x_k = w_{i_k}.$$

By Lemma 3.1, every $\mathcal{R}_{u,v,i_1,\ldots,i_k}$ is also open. Thus the set

$$\mathcal{Q}_{u,v} = \mathcal{S}_{u,v} \cup \left(\bigcup_{(i_1,\ldots,i_k) \in \mathbb{N}^k} \mathcal{R}_{u,v,i_1,\ldots,i_k} \right)$$

is open and hence the set

$$\mathcal{T}_0 = \bigcap_{u,v \in F_k} \mathcal{Q}_{u,v}$$

is a G_δ set. It is easy to see that $\mathcal{T}_0$ has the property:

T: *For every $(G, (x_1, \ldots, x_k)) \in \mathcal{T}_0$, every 2-generated subgroup of G is either cyclic of order dividing n or coincides with G.*

Let

$$\mathcal{T} = \mathcal{T}_0 \cap \mathcal{I} \cap \mathcal{S},$$

where $\mathcal{I}$ is the set of all infinite groups and $\mathcal{S}$ is the set of all simple groups in $\mathcal{G}_k$. Then $\mathcal{T}$ is a G_δ subset of $\mathcal{G}_k$. We want to show that $\mathcal{T}$ is exactly the subset of all (marked k-generated) Tarski Monsters satisfying the identity $x^n = 1$.

Indeed suppose $(G, (x_1, \ldots, x_k)) \in \mathcal{T}$. Let H be a proper subgroup of G. According to **T** every 2-generated subgroup of H is cyclic of order dividing n. This obviously implies that H itself is cyclic of order dividing n. Note also that G is infinite, simple, and satisfies $x^n = 1$ by the definition of $\mathcal{T}$. Conversely, it is easy to see that every Tarski Monster satisfying the identity $x^n = 1$ belongs to $\mathcal{T}$.

Finally to prove (f) it suffices to note that the subset of $\mathcal{G}_k$ consisting of groups with 2 conjugacy classes can be defined by the $\forall\exists$-formula

$$\forall x \,\forall y \,\exists t \,(x = 1 \,\vee\, y = 1 \,\vee\, t^{-1}xt = y).$$

Now applying part (b) of Proposition 3.2 finishes the proof. □

4 Topologizable Tarski Monsters

Our proof of Theorem 1.4 makes use of a particular variant of the general construction described in [18, Sections 25-27]. The variant used here is similar to that from [18, Section 39.2]. Below we briefly recall it and refer the reader to [18] for details.

Given a group G generated by a set X, we write "$A \equiv B$" for two words in the alphabet $X \cup X^{-1}$ if they coincide as words (i.e., letter-by-letter) and "$A = B$ in G" if A and B represent the same elements of G; by abuse of notation we identify words in $X \cup X^{-1}$ and elements represented by them. As in [18], given a word A in some alphabet, $|A|$ denotes its length.

The general construction in [18, Sections 25-27] uses a sequence of fixed positive small parameters

$$\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon,\ \zeta,\ \eta,\ \iota.$$

The exact relations between the parameters are described by a system of inequalities, which can be maid consistent by choosing each parameter in this sequence to be sufficiently small as compared to all previous parameters. In [18] and below, this way of ensuring consistency is referred to as the *lowest parameter principle* (see [18, Section 15.1]). Below we will use the following auxiliary parameters (which are assumed to be integers):

$$h = \delta^{-1},\ d = \eta^{-1},\ n = \iota^{-1}.$$

We also fix a sufficiently large odd $n_0 \in \mathbb{N}$ satisfying

$$n_0 > \max\left\{(h+1)n,\ \frac{h(d+n+2h-2)}{1-\alpha}\right\}. \tag{6}$$

Remark 4.1. Our notation in this section is borrowed from [18] and is different from the notation in the introduction: the exponent denoted by n in Theorem 1.4 is denoted by n_0 here.

Given a subset $J \subseteq \mathbb{N}$, we construct groups $G(i, J)$ by induction on $i \in \mathbb{N}$ as follows. Let $G(0, J) = F(a_1, a_2)$ be the free group with basis $\{a_1, a_2\}$. Suppose now that

$$G(i-1, J) = \langle a_1, a_2 \mid \mathcal{R}_{i-1}\rangle$$

is already constructed for some $i \geq 1$, and that for each $1 \leq j \leq i-1$ we have already defined a set $\mathcal{X}_j$ of words of length j in $\{a^{\pm 1}, b^{\pm 1}\}$ called *periods of rank* j.

The set of periods of rank i, $\mathcal{X}_i$, is defined to be a maximal set of words of length i in the alphabet $\{a^{\pm 1}, b^{\pm 1}\}$ such that no $A \in \mathcal{X}_i$ is conjugate to a power of a word of length $< i$ in the

group $G(i-1,J)$, and if A is conjugate to B or B^{-1} in $G(i-1,J)$ for some $A,B \in \mathcal{X}_i$ then $A \equiv B$.

The group $G(i,J)$ is obtained from $G(i-1,J)$ by adding a set of relations $\mathcal{S}_i$ constructed as follows. First for each period $A \in \mathcal{X}_i$, $\mathcal{S}_i$ contains the relation

$$A^{n_0} = 1 \tag{7}$$

called a *relation of the first type of rank* i.

If $i \notin J$, no other relations are included in $\mathcal{S}_i$. If $i \in J$, then for each $A \in \mathcal{X}_i$ we fix some maximal set of words $\mathcal{Y}_A$ such that:

(a) For any $T \in \mathcal{Y}_A$, we have $1 \leq |T| \leq d|A|$;

(b) Every double coset $\langle A\rangle g\langle A\rangle$ in $G(i-1,J)$ contains at most one word from $\mathcal{Y}_A$ and this word has minimal length among all words representing elements of $\langle A\rangle g\langle A\rangle$ in $G(i-1,J)$.

If $a_1 \notin \langle A\rangle$ in $G(i-1)$, then for every $T \in \mathcal{Y}_A$ such that $T \notin \langle A\rangle a_1\langle A\rangle$ in $G(i-1,J)$, we add the relation

$$a_1 A^n T A^{n+2} \dots T A^{n+2h-2} = 1 \tag{8}$$

to the set $\mathcal{S}_i$. Further if $a_2 \notin \langle A\rangle \cup \langle A\rangle a_1\langle A\rangle$ and $T \notin \langle A\rangle a_2\langle A\rangle$ in $G(i-1,J)$, then we also add the relation

$$a_2 A^{n+1} T A^{n+3} \dots T A^{n+2h-1} = 1 \tag{9}$$

to $\mathcal{S}_i$. These relations are called *relations of the second type of rank* i.

Finally we define

$$G(i,J) = \langle a_1, a_2 \mid \mathcal{R}_{i-1} \cup \mathcal{S}_i\rangle.$$

Note that there is some freedom in choosing periods in every rank and sets $\mathcal{Y}_A$. We additionally require our construction to be *uniform* in the following sense: if $I \cap [1,r] = J \cap [1,r]$ for some $r \in \mathbb{N}$, then the sets of periods and the corresponding sets $\mathcal{Y}_A$ in $G(i,I)$ and $G(i,J)$ coincide for all $1 \leq i \leq r$. In particular, $G(i,I)$ and $G(i,J)$ have the same relations for all $1 \leq i \leq r$.

Let $G(\infty,J)$ denote the limit group of the sequence $G(0,J) \to G(1,J) \to \dots$. That is,

$$G(\infty,J) = \left\langle a_1, a_2 \;\middle|\; \bigcup_{i=1}^{\infty} \mathcal{S}_i \right\rangle.$$

The presentations of $G(i,J)$, $i \in \mathbb{N} \cup \{\infty\}$, constructed above will be called *canonical*.

Remark 4.2. In our notation, the groups $G(i,j)$ constructed in [18, Section 39.2] are exactly $G(i,\{j+1,j+2,...\})$.

We will need analogues of Lemma 39.5 and Lemma 39.6 from [18]. Recall that the condition R is a technical condition which allows to apply the techniques developed in [18, Sections 25-27]. For the definition, we refer to [18, Section 25].

Lemma 4.3. (a) *For every $i \in \mathbb{N}$ and $J \subseteq \mathbb{N}$, the presentation of the group $G(i,J)$ constructed as above satisfies the condition R.*

(b) *For every $J \subseteq \mathbb{N}$, the group $G(\infty, J)$ is infinite and torsion of exponent n_0.*

(c) *If J contains all but finitely many natural numbers, then every proper subgroup of $G(\infty, J)$ is cyclic of order dividing n_0.*

Proof. The proof of the first statement almost coincides with the proof of Lemma 27.2 in [18]. The only difference is that in our construction we choose n_0 to satisfy (6), while in [18] one takes n_0 such that $n = [(h+1)^{-1}n_0]$. However the latter equality is not essential for the proof of Lemma 27.2. What is really used there is the inequality $(h+1)n \leq n_0$ (see the last line of the proof), which follows from (6).

Now part (a) allows us to apply Theorems 26.1 and 26.2 from [18], which yield (b). Finally the proof of (c) repeats the proof of [18, Lemma 39.6] verbatim after replacing $G(\infty, j)$ with $G(\infty, J)$, and j with $\max(\mathbb{N} \setminus J)$. The key point here is that all relations of the second type of rank $> \max(\mathbb{N} \setminus J)$ are imposed in $G(\infty, J)$. □

In the next lemma, we could replace "arbitrary large" with "every". However the weaker statement is sufficient for our goals.

Lemma 4.4. *For any $J \subseteq \mathbb{N}$, there exist periods of arbitrary large rank. That is, for every $r \in \mathbb{N}$, the set of periods $\mathcal{X}_i$ is non-empty for some $i > r$.*

Proof. We repeat the main argument from the proof of [18, Theorem 19.3] with obvious changes. Fix some $r \in \mathbb{N}$. By [18, Lemma 4.6] there exists a 6-aperiodic word X in the alphabet $\{a_1, a_2\}$ of length at least $20r$. Assume first that $X^{n_0} = 1$ in $G(r, J)$. Arguing as in the second paragraph of the proof of [18, Theorem 19.1] (and replacing the reference to [18, Theorem 16.2] there with the reference to [18, Theorem 22.2]) we conclude that the cyclic word X^{n_0} contains a subword of the form A^{20} for some non-trivial A of length at most r. Since the length of X is greater than $20r$, this contradicts the assumption that X is 6-aperiodic. This contradiction shows that $X^{n_0} \neq 1$ in $G(r, J)$. In particular, we have $G(\infty, J) \neq G(r, J)$ as $X^{n_0} = 1$ in $G(\infty, J)$ by part (b) of Lemma 4.3. Therefore periods of rank $> r$ exist.

□

Every group $G(i, J)$ comes with a natural generating set, namely the image of $\{a_1, a_2\}$ under the natural homomorphism $F_2 \to G(i, J)$. By abuse of notation we denote the image of $\{a_1, a_2\}$ in $G(i, J)$, $i \in \mathbb{N} \cup \{\infty\}$, by $\{a_1, a_2\}$ as well. In what follows we say that a homomorphism $\varepsilon\colon G(\infty, I) \to G(\infty, J)$ is *natural* if $\phi(a_1) = a_1$ and $\phi(a_2) = a_2$.

Lemma 4.5. *Let $J \subseteq \mathbb{N}$ and let $I = J \cap [1, r]$ for some $r \in \mathbb{N}$. Then the following hold:*

(a) *There exists a natural homomorphism $\varepsilon\colon G(\infty, I) \to G(\infty, J)$.*

(b) $\operatorname{Ker} \varepsilon$ *does not contain nontrivial elements of $G(\infty, I)$ of length $\leq r$ with respect to the generating set $\{a_1, a_2\}$.*

Proof. We first note that claim (a) is not obvious as, in general, the set of defining relations in the canonical presentation of $G(\infty, I)$ is not a subset of the set of relations in the canonical presentation of $G(\infty, J)$. However it is possible to construct other presentations of $G(\infty, I)$ and $G(\infty, J)$ for which this is the case.

Let $\mathcal{R}_I$ and $\mathcal{R}_J$ be the sets of relations of the second type in the canonical presentations of $G(\infty, I)$ and $G(\infty, J)$, respectively. By uniformness of our construction, we have $\mathcal{R}_I \subseteq \mathcal{R}_J$. Since both $G(\infty, I)$ and $G(\infty, J)$ are torsion of exponent n_0 by part (b) of Lemma 4.3 and all relations of the first type have the form $X^{n_0} = 1$ for some word X in the alphabet $\{a_1^{\pm 1}, a_2^{\pm}\}$, we can represent the groups $G(\infty, I)$ and $G(\infty, J)$ as follows:

$$G(\infty, I) = \langle a_1, a_2 \mid \mathcal{R}_I,\ X^{n_0} = 1\ \forall X \rangle$$

and

$$G(\infty, J) = \langle a_1, a_2 \mid \mathcal{R}_J,\ X^{n_0} = 1\ \forall X \rangle,$$

where the relations $X^{n_0} = 1$ are imposed for all words X in $\{a_1^{\pm 1}, a_2^{\pm}\}$. Now part (a) of the lemma becomes obvious.

Further part (a) of Lemma 4.3 allows us to apply Lemma 23.16 from [18], which implies that every nontrivial element from $\operatorname{Ker} \varepsilon$ has length at least $(1-\alpha)$ times the minimal possible length of a relator of rank $> r$. It is easy to see from (7)-(9), that the length of every relator of rank $> r$ is at least $(r+1)\min\{n_0, (2h-1)n\} > rn$. By the lowest parameter principle we can assume that $(1-\alpha)n > 1$. Hence every nontrivial element from $\operatorname{Ker} \varepsilon$ has length at least r. □

In what follows we think of $G(\infty, J)$ (or, more precisely, $(G(\infty, J), \{a_1, a_2\})$) as an element of $\mathcal{G}_2$. Let $\mathcal{T}$ be the subspace of $\mathcal{G}_2$ consisting of $G(\infty, J)$ for all $J \subseteq \mathbb{N}$. To apply the Baire Theorem to $\mathcal{T}$ we need to know that $\mathcal{T}$ is complete as a metric space. We will prove this by showing that $\mathcal{T}$ is a continuous image of the Cantor set. Recall that the Cantor set C can be identified with $2^{\mathbb{N}}$, where the distance between any two distinct subsets $I, J \subseteq \mathbb{N}$ is defined by

$$\mathrm{d}(I, J) = \frac{1}{\min(I \vartriangle J)}.$$

Corollary 4.6. *The map from the Cantor set C to $\mathcal{T}$ defined by $J \mapsto (G(\infty, J), (a_1, a_2))$ is Lipschitz. In particular, this map is continuous and $\mathcal{T}$ is compact.*

Proof. Let $I, J \subseteq \mathbb{N}$. Suppose now that $\mathrm{d}(I, J) = 1/r$ for some $r \geq 1$ in C. Let $K = I \cap [1, r-1] = J \cap [1, r-1]$. By part (b) of Lemma 4.5, we have $\mathrm{d}(G(\infty, I), G(\infty, K)) \leq 1/r$ and $\mathrm{d}(G(\infty, J), G(\infty, K)) \leq 1/r$. Since d is an ultrametric, we obtain

$$\mathrm{d}(G(\infty, I), G(\infty, J)) \leq \max\{\mathrm{d}(G(\infty, I), G(\infty, K)), \mathrm{d}(G(\infty, J), G(\infty, K))\} \leq 1/r = \mathrm{d}(I, J).$$

Thus the map $C \to \mathcal{T}$ is 1-Lipschitz. □

Our next goal is to show that $\mathcal{T}$ contains a dense subset of topologizable groups. We begin with an auxiliary result.

Lemma 4.7. *Let I be a finite subset of $\mathbb{N}$. Then for every non-trivial element $g \in G(\infty, I)$, there exists a non-trivial normal subgroup $N \lhd G(\infty, I)$ such that $g \notin N$.*

Proof. Let l denote the word length of the element g with respect to the generating set $\{a_1, a_2\}$. By Lemma 4.4, there exists a period A of some rank

$$i > \max\{l, \max I\}, \tag{10}$$

Since $G(\infty, I)$ is infinite by part (b) of Lemma 4.3, we can additionally assume that balls of radius i in $G(\infty, I)$ contain more than n_0^2 elements.

Note that the double coset $\langle A\rangle a_1 \langle A\rangle$ in $G(\infty, I)$ contains at most n_0^2 elements as $A^{n_0} = 1$ in $G(\infty, I)$. Therefore, by our choice of i, there exists a word T of length $1 \le |T| \le i < di$ such that T does not belong to $\langle A\rangle a_1 \langle A\rangle$ in $G(\infty, I)$. Hence T does not belong to $\langle A\rangle a_1 \langle A\rangle$ in $G(i-1, I)$. Replacing T with the shortest word among all words representing elements of the double coset $\langle A\rangle T \langle A\rangle \le G(i-1, I)$ if necessary, we can assume that $T \in \mathcal{Y}_A$.

Let now $J = I \cup \{i\}$. By (10), Lemma 4.5 applies to I and J with $r = i - 1 \ge l$. Let N be the kernel of the natural homomorphism $G(\infty, I) \to G(\infty, J)$. Then by part (b) of Lemma 4.5 we have $g \notin N$.

It remains to show that N is nontrivial. To this end, we will show that

$$1 \ne a_1 A^n T A^{n+2} \dots T A^{n+2h-2} \in N$$

in $G(\infty, I)$. Indeed $a_1 A^n T A^{n+2} \dots T A^{n+2h-2} \in N$ by the construction of $G(\infty, J)$. Suppose that $a_1 A^n T A^{n+2} \dots T A^{n+2h-2} = 1$ in $G(I, \infty)$. Let Δ be the corresponding reduced disk diagram over $G(I, \infty)$. Then Δ is a B-map by [18, Lemma 26.5] and part (a) of Lemma 4.3.

Note that Δ does not contain faces of rank $\ge i$. Indeed if such faces existed, they would correspond to relations of the first type as there are no relations of the second type of rank $\ge i$ in $G(I, \infty)$. However by our choice of n_0 this is impossible since these faces are "too large"; more precisely, by [18, Lemma 23.16], the perimeter of each face in Δ is at most

$$\frac{|\partial \Delta|}{1-\alpha} \le \frac{hi(d+n+2h-2)}{1-\alpha} < n_0 i$$

(see (6)), while the length of every relation of the first type of rank $\ge i$ is at least $n_0 i$.

Thus Δ is a diagram over $G(i-1, I)$. Since $G(i-1, I) = G(i-1, J)$, Δ is also a diagram over $G(i-1, J)$. Hence the relation $a_1 A^n T A^{n+2} \dots T A^{n+2h-2} = 1$ can be derived from relations of rank $< i$ in $G(\infty, J)$. This contradicts [18, Corollary 25.1], which guarantees that the relations of the canonical presentation of $G(\infty, J)$ are independent. The contradiction shows that $a_1 A^n T A^{n+2} \dots T A^{n+2h-2} \ne 1$ in $G(I, \infty)$ and therefore N is non-trivial. □

Corollary 4.8. *Let J be a finite subset of $\mathbb{N}$. Then $G(\infty, J)$ is topologizable.*

Proof. It suffices to construct a sequence of infinite normal subgroups

$$N_1 \rhd N_2 \rhd \dots \tag{11}$$

of $G(\infty, J)$ with trivial intersection. Then taking $\{N_i\}_{i\in\mathbb{N}}$ as the base of neighborhoods of 1, we obtain a group topology on $G(\infty, J)$ which is Hausdorff as $\bigcap_{i\in\mathbb{N}} N_i = \{1\}$ and is non-discrete as every N_i is infinite.

To this end, we first note that every non-trivial normal subgroup $M \lhd G(\infty, J)$ is infinite. Indeed otherwise the centralizer $C_{G(\infty,J)}(M)$ has finite index in $G(\infty, J)$. By [18, Theorem 26.5] the centralizer of every element in $G(\infty, J)$ is cyclic. Since $G(\infty, J)$ is torsion by part (b) of Lemma 4.3 we obtain that $C_{G(\infty,J)}(M)$ is finite and hence so is $G(\infty, J)$. However this contradicts part (b) of Lemma 4.3.

Now we construct the desired sequence (11) by induction. Let $G(\infty, J) = \{1, g_1, g_2, \ldots\}$. By Lemma 4.7 we can find a non-trivial subgroup $N_1 \lhd G(\infty, J)$ that does not contain g_1. Suppose that N_j is already constructed for some $j \geq 1$ and $\{g_1, \ldots, g_j\} \cap N_j = \varnothing$. Applying Lemma 4.7 again, we can find a non-trivial subgroup $N \lhd G(\infty, J)$ such that $g_{j+1} \neq N$. Let $N_{j+1} = [N_j, N]$. Obviously $\{g_1, \ldots, g_{j+1}\} \cap N_{j+1} = \varnothing$. Note also that N_{j+1} is nontrivial. Indeed otherwise $N_j \leq C_{G(\infty,J)}(N)$ and arguing as in the previous paragraph we obtain that N_j is finite; however this contradicts the fact that every non-trivial normal subgroup of G is infinite. This completes the inductive step. Obviously $\bigcap_{i\in\mathbb{N}} N_i = \{1\}$ and thus the lemma is proved. $\square$

Remark 4.9. If D is a G_δ subset of a topological space X and Y is a subspace of X, then $D \cap Y$ is a G_δ subset of Y. This observation will be used several times below.

Proof of Theorem 1.4. Note that the set of finite subsets of $\mathbb{N}$ is dense in the Cantor set C. Hence its image is dense in $\mathcal{T}$ by Corollary 4.6. Using Lemma 4.8 we obtain that $\mathcal{T}$ contains a dense subset of topologizable groups. Then by Proposition 3.4 and Remark 4.9 we conclude that the property of being topologizable is generic in $\mathcal{T}$.

Further the set of all cofinite subsets of $\mathbb{N}$ is also dense in the Cantor set. Using Lemma 4.3 and arguing as in the previous paragraph, we obtain that the property of being a Tarski Monster (of exponent n_0) is generic in $\mathcal{T}$.

Since $\mathcal{T}$ is compact, we can apply the Baire Category Theorem, which implies that the property of being a topologizable Tarski Monster (of exponent n_0) is also generic in $\mathcal{T}$. In particular, such groups exist. $\square$

5 Further examples

One can produce many other examples of "exotic" topologizable groups using the fact that most limits of "hyperbolic-like" groups are topologizable. More precisely, let $\mathcal{H}_k \subseteq \mathcal{G}_k$ denote the set of all (marked) groups from $\mathcal{G}_k$ containing non-degenerate hyperbolically embedded subgroups. Recall that this class is very wide and contains all non-elementary hyperbolic groups, non-elementary relatively hyperbolic groups with proper peripheral subgroups, as well as many other groups acting on hyperbolic spaces. For the precise definition and more examples we refer to [5]. We begin by proving that every $G \in \mathcal{H}_k$ is topologizable.

Lemma 5.1. *Every $G \in \mathcal{H}_k$ is topologizable.*

Proof. The proof relies heavily on results of [5]; for the particular cases of hyperbolic and relatively hyperbolic groups one could alternatively use [19] or [23].

Let $G \in \mathcal{H}_k$. By [5, Theorem 2.23], there exists a hyperbolically embedded subgroup $H \hookrightarrow_h G$ such that $H \cong \mathbb{Z} \times K$, where K is a finite group. In particular, H contains an infinite chain of infinite normal (in H) subgroups $N_1 \rhd N_2 \rhd \ldots$ with trivial intersection. Let M_i denote the normal closure of N_i in G. Then the group-theoretic Dehn surgery theorem (see [5, Theorem 2.25 (c)]) implies that $\bigcap_{i\in\mathbb{N}} M_i = \{1\}$. Now we can use the chain $M_1 \rhd M_2 \rhd \ldots$ to define a Hausdorff topology on G, taking $\{M_i \mid i \in \mathbb{N}\}$ as the base of neighborhoods at 1. Since every N_i is infinite, so is M_i and hence the topology is non-discrete. $\square$

Using chains of normal subgroups $N_1 \rhd N_2 \rhd \ldots$ as bases of neighborhoods is a fairly standard approach to defining a topology on a given group G. It is interesting to ask whether one can topologize a "hyperbolic-like" group in an essentially different way. The question can be formalized as follows: *Under what conditions does a group $G \in \mathcal{H}_k$ admit a topology with respect to which it is topologically simple?*

Groups with proper infinite hyperbolically embedded subgroups are very far from being abstractly simple [5]. Of course, G is not topologically simple in the topology defined in the proof of Lemma 5.1 as well, since every N_i is closed. However, we conjecture the following.

Conjecture 5.2. *Suppose that a group $G \in \mathcal{H}_k$ has no non-trivial finite normal subgroups. Then G admits a topology with respect to which it is topologically simple.*

Note that the absence of non-trivial finite normal subgroups is necessary as finite subgroups are always closed.

Conjecture 5.2 holds for hyperbolic groups. Indeed, Chaynikov [3] proved that every non-elementary hyperbolic group G without non-trivial finite normal subgroups admits a faithful action on $\mathbb{N}$ which is k-transitive for every $k \in \mathbb{N}$. This action defines a dense embedding $G \to S(\mathbb{N})$, where $S(\mathbb{N})$ is the group of all permutations of $\mathbb{N}$ endowed with the topology of pointwise convergence. Let $A_{fin}(\mathbb{N}) = \bigcup_{n\in\mathbb{N}} A_n$ and $S_{fin}(\mathbb{N}) = \bigcup_{n\in\mathbb{N}} S_n$, where A_n and S_n are the groups of even permutations and all permutations of $\{1, \ldots, n\}$, respectively, naturally embedded in $S(\mathbb{N})$. Then $A_{fin}(\mathbb{N})$ and $S_{fin}(\mathbb{N})$ are the only proper non-trivial normal subgroups of $S(\mathbb{N})$ (see [7, Theorem 8.1A]). Obviously both of them are dense and hence $S(\mathbb{N})$ is topologically simple. Now using the fact that the image of G is dense in $S(\mathbb{N})$ it is straightforward to verify that G is topologically simple with respect to the topology induced by the embedding. In seems plausible that the Chaynikov's result can be generalized to groups from $\mathcal{H}_k$, which would imply Conjecture 5.2 in the full generality.

Proof of Theorem 1.6. The theorem obviously follows from Lemma 5.1, part (a) of Proposition 3.4, and Remark 4.9. $\square$

To illustrate usefulness of Theorem 1.6, we outline here the proof of the existence of a topologizable groups with 2 conjugacy classes. First examples of groups with 2 conjugacy classes other than $\mathbb{Z}/2\mathbb{Z}$ were constructed by Higman, B.H. Neumann and H. Neumann in 1949; first finitely generated examples were constructed by the third author in [22]. Motivated by

the recent study of groups with the Rokhlin property, (i.e., topological groups with a dense conjugacy class) Glassner and Weiss ask in [8] whether there exist topological analogues of these constructions. Specifically, they ask whether there exists a non-discrete locally compact topological group with 2 conjugacy classes. Our approach allows to construct a non-discrete group with 2 conjugacy classes; local compactness can not be ensured by our methods although our group will be compactly generated (and even finitely generated in the abstract sense).

To construct a topologizable group with 2 conjugacy classes we first recall that groups with exactly two conjugacy classes form a G_δ subset of $\mathcal{G}_k$ by part (f) of Proposition 3.4. Further let $k \geq 2$ and let $\mathcal{RH}$ denote the subset of all groups from $\mathcal{G}_k$ that are torsion free, non-cyclic, and hyperbolic with respect to a collection of proper subgroups. Then the technique developed in [22] can be used to show that for every $G \in \mathcal{RH}$ and every $\varepsilon > 0$, there exists a (marked k-generated) group Q in $\overline{\mathcal{RH}}$ such that $\mathrm{d}(G, Q) < \varepsilon$ in $\mathcal{G}_k$ and Q has exactly 2 conjugacy classes. Thus $\overline{\mathcal{RH}}$ contains a dense G_δ subset of groups with 2 conjugacy classes. Combining this with Theorem 1.6, we obtain that a generic group in $\overline{\mathcal{RH}}$ is topologizable and all its non-trivial elements are conjugate.

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